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Mathematics

On the spectral geometry of spaces with cone-like singularities

(Hodge theory/asymptotic expansion/heat kernel)

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ABSTRACT I describe an extension of a portion of the theory of the Laplace operator on compact riemannian manifolds to certain spaces with singularities. Although this approach can be extended to include quite general spaces, this paper will confine itself to the case of manifolds with cone-like singularities. These singularities are geometrically the simplest possible, but they already serve to illustrate new phenomena that are typical of the more general situation. Moreover, by inductive arguments, the study of simplicial complexes whose simplices have constant curvature and totally geodesic faces (e.g., *p.l.* manifolds) can in large measure be reduced to the study of cone-like singularities.

1. Introduction

The general program can be described roughly as follows. Let X^n be a metric space such that, for some closed subset Σ^m, $X^n - \Sigma^m$ is a smooth dense (incomplete) riemannian manifold in the induced metric. We study the L^2 theory of the Laplace operator on the manifold $X^n - \Sigma^m$. Although $X^n - \Sigma^m$ is open, for a large class of spaces the compactness of X^n forces the most fundamental features of the theory on compact manifolds to continue to hold for $X^n - \Sigma^m$. However, when the detailed consequences of these are examined in the more general setting, certain qualitative aspects emerge that are not present or, one might say, are disguised in the nonsingular case. As a striking example, the cohomology theory which is naturally related to Hodge theory is (still) L^2-cohomology. But, although the L^2-cohomology groups of $X - \Sigma$ are topological invariants of X, they do not coincide with the usual simplicial cohomology groups unless X is a rational homology manifold.

The principle technique on which the results of this paper are based is a functional calculus in which the method of separation of variables is used systematically in combination with the functional calculus for the Laplace operator on riemannian manifolds (see section 4). The main point here is to reduce local analysis near the vertex of a cone to global analysis on the base. Analogues of this technique are useful in essentially all problems involving separation of variables.

2. L^2-Cohomology

Let N^m be a riemannian manifold with metric g and possibly nonempty boundary ∂N^m. By the *metric cone* $C(N)$ on N, we mean the space $(0, \infty) \times N$, equipped with the metric

$$dr \otimes dr + r^2 g. \qquad [2.1]$$

Set

$$C_{u,u_0}(N) = \{(r, x) \in C(N) | u < r \le u_0\}. \qquad [2.2]$$

Definition 2.1: X^{m+1} is a riemannian manifold with conelike singularities if there exist $p_j \in X^{m+1}$, $j = 1 \ldots K$ such that $X^{m+1} - \cup_1^N p_j$ is a smooth open riemannian manifold (possibly with boundary) and each p_j has a neighborhood U_j such that $U_j - p_j$ is isometric to $C_{0,u_j}(N_j^m)$ for some u_j and N_j^m (which might not be connected). Set $C_{0,1}^*(N) = puC_{0,1}(N)$.

Because the general case is no harder, to simplify the exposition we now assume that $K = 1$ and $u_j \ge 1$. We write $X = C_{0,1}^*(N) \cup M$ where $\partial M = N$ and the union is along the boundary. Thus, if $\partial N = \phi$, X can be pictured as the surface of an ice cream cone, and if $\partial N \ne \phi$, as the ice cream cone itself. Of course, X is not homeomorphic to a manifold in general.

There are two examples in which no singularity is actually present: $N^m = S_1^m$, the unit sphere; and $N^m = H_1^m$, the unit hemisphere. In these cases, $C(S_1^m) = R^{m+1}$, $C(H_1^m) = R^+ \times R^m$, and X is smooth and flat near p. For $N^0 = (H_1^0) = q$, the completed cone $p \cup C(N) = C^*(N) = R^{\ge 0}$ has a boundary, even though $\partial q = \phi$. Thus, to do analysis on $C(q)$ one must first choose boundary conditions. It will be shown below that the same situation obtains whenever $H^k(N^{2k}, R) \ne 0$, even though in higher dimensions p does not have the appearance of a boundary point.

The necessity to choose boundary conditions already arises from Poincaré duality considerations in L^2-cohomology. To shorten the exposition we assume that $\partial N = \partial X = \phi$. All our results have straightforward generalizations to the case of *absolute* or *relative* boundary conditions (see ref. 1). These will be described elsewhere. In the case $m = 2k$, fix a subspace $V_a \subset H^k(N^{2k}, \mathbf{R})$. By an *i-form on X* we mean an i-form on the smooth open riemannian manifold $X - p$. Let $\hat{C}^i$ denote the space of smooth closed i-forms ϕ that are in L^2 and that, in case $m = 2k$, satisfy the boundary conditions of Eq. **3.17** below. Let $\hat{E}^i \subset \hat{C}^i$ denote those forms such that $\phi = d\psi$ for some $\psi \in L^2$ (satisfying Eq. **3.17** if $m = 2k$).

Definition 2.2: $\hat{H}^i(X)$ the i-th L^2-cohomology group of X is the vector space $\hat{C}^i/\hat{E}^i$.

Let

$$H^{i-1}(N) \xrightarrow{\delta} H^i(M, N) \xrightarrow{i} H^i(M) \xrightarrow{j} H^i(N) \rightarrow \qquad [2.3]$$

denote the exact sequence of the pair M, N. Then an L^2 version of the Poincaré lemma and Mayer Vietoris sequence yields.

THEOREM 2.1.

$m = 2k - 1$

$$\hat{H}^k(X) = \begin{cases} H^i(M) & i < k \\ i(H^k(M, N)) \subset H^k(M) & i = k \\ H^i(M, N) & i > k \end{cases}$$

$m = 2k$

$$\hat{H}^i(X) = \begin{cases} \hat{H}^i(M) & i < k \\ j^{-1}(V_a) \subset H^k(M) & i = k \\ H^{k+1}(M, N)/\delta(V_a) & i = k + 1 \\ \hat{H}^i(M, N) & i < k + 1 \end{cases}$$

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COROLLARY 2.1 (*Poincaré duality*). *If* X *is orientable and* $m = 2k - 1$, $\hat{H}^i(X) \simeq \hat{H}^{m+1-i}(X)$. If $m = 2k$, (sig N = 0) *and* V_a *is a maximal self-annihilating subspace under the cup product pairing, then* $\hat{H}^i(X)$.

The groups $\hat{H}^i(X)$ are topological invariants of X but do not give the usual cohomology unless X is a rational homology manifold. Rather they are isomorphic to the so called "middle groups" defined via complexes of special chains by Goresky and MacPherson (2) and also by Morgan. This connection was conjectured by Dennis Sullivan (personal communication) after he saw *Theorem 2.1*.

3. Hodge theory

Because we now want to construct a theory that generalizes Hodge theory, we will have to build in the requirement that, if a form θ is in the domain of the Laplacian Δ, then θ, $\Delta\theta$, $d\theta$, and $\delta\theta$ are all in L^2. If we omit the requirement $d\theta$, $\delta\theta \in L^2$, then we would sometimes have to admit harmonic forms that were not closed and co-closed. Conversely, the requirement $d\theta$, $\delta\theta \in L^2$ makes the theory unique [modulo the choice of boundary conditions in case $H^k(N^{2k},R) \neq 0$]. We will begin, therefore, by examining what it means for a harmonic form θ to have the property that θ, $d\theta$, $\delta\theta \in L^2$.

Let

$$\theta = g(r)\phi + f(r)dr \wedge \omega \quad [3.1]$$

be a smooth i-form on $C(N)$. Note that

$$*\theta = (-1)^i r^{m-2i} g dr \wedge \tilde{*}\phi + r^{m-2i+2} f \tilde{*}\omega. \quad [3.2]$$

Also, a straightforward calculation shows that

$$\begin{aligned}\Delta\theta = {} & [-g'' - (m-2i)r^{-1}g']\phi + r^{-2}g\tilde{\Delta}\phi \\ & + 2r^{-3}g dr \wedge \tilde{\delta}\phi \\ & + [-f'' - (m-2i)r^{-1}f' + (m-2i)r^{-2}f]dr \wedge \omega \\ & + r^{-2}f dr \wedge \tilde{\Delta}\omega - 2r^{-1}f d\omega. \end{aligned} \quad [3.3]$$

Here $\tilde{*}$, $\tilde{\delta}$, $\tilde{\Delta}$ denote the intrinsic operations on N^m, which we now assume to be compact. Eq. 3.3 leads to equations of Bessel type, with a regular singularity at $r = 0$, for the eigenforms of Δ. Let

$$\alpha_i = (1 + 2i - m)/2 \quad [3.4]$$

$$\nu(i) = \sqrt{\alpha_i^2 + \mu} \quad [3.5]$$

$$a^{\pm}(i) = \alpha_i \pm \nu(i). \quad [3.6]$$

Then a standard argument shows that the harmonic i-forms on $C_{u,u_0}(N)$ can be written as (convergent) sums of forms of the following four types:

$$r^{a^{\pm}(i)}\phi \quad [3.7]$$

$$r^{a^{\pm}(i-1)}d\omega + a^{\pm}(i-1)r^{a^{\pm}(i-1)-1}dr \wedge \omega \quad [3.8]$$

$$a^{\pm}(i-1)r^{a^{\pm}(i-1)}\rho + r^{a^{\pm}(i-1)-1}dr \wedge \tilde{\delta}\rho \quad [3.9]$$

$$r^{a^{\pm}(i-2)+1}dr \wedge \psi \quad [3.10]$$

where ϕ, ω and ρ, ψ are co-closed and closed eigenforms of $\tilde{\Delta}$, respectively, with eigenvalue μ. The eigenforms of Δ with eigenvalue $\lambda^2 \neq 0$ have similar representations involving Bessel functions, $J_\nu(\lambda r)$. If $\nu = 0$, a logarithmic solution must also be introduced. If ϕ, ω and ρ, ψ are co-exact and exact, the eigenfunctions corresponding to 3.7 to 3.10 will be called of types 1–4, respectively. Note that types 1 and 3 are co-exact and types 2 and 4 are exact. In case ϕ, ω, ρ, ψ are harmonic, the distinction between types 1 and 3 and also between types 2 and 4 disappears.

By using Eq. 3.2, a straightforward examination of 3.7 to 3.10 shows that the square of the pointwise norm of $\theta^{\pm}$ is asymptotic to a constant times

$$\begin{matrix} r^{1-m\pm2\nu(i)} & \nu(i) > 0 \\ r^{1-m}, r^{1-m}\log^2 r & \nu(i) = 0 \end{matrix} \quad [3.11]$$

$$r^{-1-m\pm2\nu(i-1)} \quad [3.12]$$

$$\begin{matrix} r^{1-m\pm2\nu(i-2)} & \nu(i-2) > 0 \\ r^{1-m}, r^{1-m}\log^2 r & \nu(i-2) = 0 \end{matrix} \quad [3.13]$$

for types 1, 2 and 3, and 4, respectively. The condition that $\theta \in L^2$ is just

$$\|\theta\|^2 < Kr^{-(m+1)}. \quad [3.14]$$

Thus, it is apparent that

(*a*) $\theta^+, d\theta^+, \delta\theta^+ | C_{0,u}(N) \in L^2$

(*b*) $\theta^- | C_{0,u}(N) \in L^2$ implies θ^- is of type 1 or 4 and $\nu(i) < 1$ respectively $\nu(i-2) < 1$. Note that $\nu(i) < 1$ implies either $i = m - 1/2$ or $i = m/2 - 1, m/2$.

(*c*) $\theta^-, d\theta^-, \delta\theta^- | C_{0,u}(N) \in L^2$, in view of *b*, implies $d\theta \equiv \delta\theta \equiv 0$. Thus, $m = 2i$, $\mu = 0$ for type 1 and $m = 2i - 2$, $\mu = 0$, for type 4.

In fact, it follows from Eq. 3.3 that in case $m = 2i$, $\mu = 0$, the condition that $g(r)\phi$ be an eigenform of Δ is just

$$-g''\phi = \lambda^2 g\phi. \quad [3.15]$$

Because Eq. 3.15 is identical with the equation for special case of the half-line ($N^0 = q$), it is clear that, in order to proceed further, we will have to choose boundary conditions in this case. Fix an *orthogonal* direct sum decomposition

$$H^k(N^{2k},R) = V_a \oplus V_r \quad [3.16]$$

and let $\{\phi_i\}$, $\{\Psi_j\}$ be orthonormal bases for the corresponding spaces of $\mathcal{H}^k(N)$, the space of harmonic k-forms. We say that a k form $\theta(r,x)$ satisfies the boundary conditions $V_a \oplus V_r$ if the orthogonal projection $\pi_{\mathcal{H}^k(N)}[\theta(r,x)]$ of $\theta(r,x)|(1,N)$ satisfies

$$\pi_{\mathcal{H}^k(N)}[\theta(r,x)] = \Sigma f_i\phi_i + \Sigma g_j\psi_j. \quad [3.17]$$

where $f_i'(0) = g_j(0) = 0$. The definition for $(k+1)$ forms is similar (and is also referred to as Eq. 3.17).

THEOREM 3.1. *If* $m = 2k - 1$, *the collection of all eigen* i-*forms* ϕ, *such that* ϕ, $d\phi$, $\delta\phi \in L^2$ *determines a complete orthonormal basis of* L^2. *If* $m = 2k$, *the collection of such eigenforms satisfying boundary conditions* $V_a \oplus V_r$ *determine a complete orthonormal basis. The eigenspaces are finite dimensional and the eigenvalues satisfy* $0 \le \lambda_1 \le \lambda_2 \cdots \to \infty$.

Implicit in *Theorem 3.1* is the Hodge decomposition and the Hodge isomorphism $\hat{H}^i(X) \simeq \mathcal{H}^i$ where $\mathcal{H}^i$ is the space of harmonic i-forms satisfying the conditions of the theorem. Poincaré duality also follows (if in case $m = 2k$, $*V_a = V_r$) because $*$ maps harmonic forms to harmonic forms. $*V_a = V_r$ is equivalent to the condition of *Corollary 2.1*.

Theorem 3.1 leads to an analogue of the Bochner vanishing theorem (3) for $\hat{H}^*(X)$. This theorem illustrates a geometric interpretation of the condition $\nu \ge 1$ as a kind of non-negative curvature condition at p. Moreover, there is a generalization to the *p.l.* case which is quite interesting and which will be described elsewhere.

4. A functional calculus for $C(N)$

The results of *section* 3 follow by essentially the same arguments as in the nonsingular case (Fredholm theory applied to the compact self-adjoint Green's operator) once a parametrix has been constructed and shown to possess sufficiently good properties. The parametrix, in turn, is gotten by blending together the precise fundamental solution on the cone $C(N)$ with a standard parametrix on $X - C_{0,u_0}(X)$, for some u_0. Thus, $C(N)$ plays the role of the tangent space at p; analysis on $C(N)$ is "hard" and the subsequent globalization of this analysis to X is "soft." In case X is only asymptotically conical, a cruder parametrix can still be constructed.

In order to form functions of Δ on $C(N)$ we replace the Fourier transform, which is available only in the nonsingular case $N = S_1^m$, $C(S_1^m) = R^{m+1}$, by the Hankel transform (4) combined with the functional calculus on N (compare ref. 5, p. 179). In this section I briefly indicate a few of the most important examples of this procedure. Discussion of the exact domain of validity of the formulas will be given elsewhere. For brevity, I will also restrict myself to forms of type 1 (which includes the case of functions). The results of sections 4–6, however, depend on the formulas corresponding to **4.4** for types 2–4.

According to a functional calculus based on the Hankel inversion formula, for forms of type 1 we have the following formal representation for the kernel of the operator $f(\Delta)$.

$$f(\Delta) = (r_1r_2)^{\alpha_i} \sum_j \left[\int_0^\infty f(\lambda^2)J_{\nu_j}(\lambda r_1)J_{\nu_j}(\lambda r_2)\lambda d\lambda\right] \times \phi_j(x_1) \otimes \phi_j(x_2), \quad [\mathbf{4.1}]$$

where ϕ_j runs over an orthonormal basis of co-closed eigenforms of N, and α, ν are defined as in Eqs. **3.4** and **3.6.** In order to "sum the series" we regard the right-hand side of **4.1** as a family of functions of $\tilde{\Delta}$ (or more precisely its co-closed part) on N which is parameterized by r_1, r_2. These functions can then be synthesized out of, for example, the resolvent, heat operator, or wave operator and the asymptotic behavior of those operators can be applied to discuss the behavior of $f(\Delta)$.

Example 4.1 (Δ^{-1}, Green's operator): In this case, the Hankel transform can be avoided because $f(\Delta)$ is defined by local conditions:

$$\Delta^{-1} = (r_1 \cdot r_2)^{\alpha_i} \sum_j \frac{(r_1/r_2)^{\nu_j}}{2\nu_j} \phi_j(x_1) \otimes \phi_j(x_2) \qquad r_1 \le r_2. \quad [\mathbf{4.2}]$$

The right-hand side is a Poisson type kernel (compare ref. 6) on N evaluated at $\log(r_2/r_1)$. It can be synthesized out of the heat operator $P_{cc}e^{-t\tilde{\Delta}}$. Here, P_{cc} denotes orthogonal projection on the co-closed subspace. For $\nu_j > 0$, **4.2** then becomes

$$\frac{(r_1 \cdot r_2)^{\alpha_i}}{2\sqrt{\pi}} \int_0^\infty u^{-1/2} exp\left(-\frac{\log^2(r_2r_1)}{4u} - \alpha^2 u\right) P_{cc}e^{-\tilde{\Delta}u}du \quad [\mathbf{4.3}]$$

(compare ref. 7).

More generally, $\Gamma(s)\Delta^{-s}$ can be studied with help of the *Weber Schaftheitlin* integral. For $r_1 = r_2 = 1$ this leads to the following formula for the pointwise trace of $\Gamma(s)\Delta^{-s}$. Let b_i denote the ith Betti number of N and $\nu = \sqrt{\alpha_i^2 + \Delta}$.

$$tr[\Gamma(s)\Delta^{-s}(\Delta)]|_{r_1=r_2=1} = trP_{cc} \times \left(\frac{1}{2\sqrt{\pi}} \frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \Gamma(s - 1/2)\right)$$

$$\overset{def}{=} \frac{b_i}{2\sqrt{\pi}} \frac{\Gamma(|\alpha_i| - s + 1)}{\Gamma(|\alpha_i| + s)} \Gamma(s - 1/2) + \psi_i(s). \quad [\mathbf{4.4}]$$

$\psi_i(s)$ can be shown to be a meromorphic function whose properties closely resemble those of $\zeta_{ce}(s - 1/2)$, where $\zeta_{ce}(s)$ is the zeta function for the Laplacian on co-exact i-forms of N. Like $\zeta_{ce}(s)$, $\psi_i(s)$ depends only on the spectrum for co-exact i-forms but can be rewritten in terms of the full spectrum for j-forms, $j \le i$ (compare ref. 8). Similarly, the heat kernel and wave kernel can be studied via *Weber's second exponential integral* and the *Lipschitz Hankel integral* (see ref. 9, pp. 401, 395, and 389 for the above integrals). I do this elsewhere.

5. The asymptotic expansion for the trace of the heat kernel

By methods like those outlined in sections 3 and 4, the fundamental solution $E(t)$ for the heat equation on X can be constructed. Let $a_{j/2}(r,x)$ denote the pointwise coefficient of $t^{-(m+1)/2+j/2}$ in the asymptotic expansion of the fundamental solution $\mathscr{E}(t)$ on $C(N)$. Here, we have included half powers of t to cover the case $\partial N \neq \phi$. An easy scaling argument shows that

$$a_{j/2}(r,x) = r^{-m+j}a_{j/2}(1,x). \quad [\mathbf{5.1}]$$

In particular, setting $X_u = X - C_{0,u}(N)$, $X_1 = M$, we have

$$\int_{X_u} a_{j/2}(y) = \int_M a_{j/2}(y) + \left(\frac{1}{m+1-j} - \frac{u^{m+1-j}}{m+1-j}\right) \times \int_N a_{j/2}(1,x) \qquad j \neq m+1$$

$$\int_{X_u} a_{j/2}(y) = \int_M a_{j/2}(y) - \log u \int_N a_{j/2}(1,x) \qquad j = m+1. \quad [\mathbf{5.2}]$$

Thus, the integral over X converges for $j < m + 1$ and, in general, we can define its *finite part* by

$$p.f.\ a_{j/2} = p.f. \int_X a_{j/2} = \int_M a_{j/2} + \frac{1}{(m+1)/2 - j/2} \int_N a_{j/2}(1,x) \qquad j \neq m+1 \quad [\mathbf{5.3}]$$

$$p.f.\ a_{j/2} = p.f. \int_X a_{j/2} = \int_M a_{j/2} \qquad j = m+1$$

Set

$$\mu_K(u) = \int_N tr\ \mathscr{E}(1,x,1,x,t) - \sum_{j=0}^K \left[\int_N a_{j/2}(1,x)\right] u^{-(m+1)/2+j/2}. \quad [\mathbf{5.4}]$$

THEOREM 5.1.

(i) $\int_{C_{0,1}(N)} tr\ \mathscr{E}_i(t) = 1/2 \int_t^\infty u^{-1} \int_N tr\ \mathscr{E}_i(1, x, 1, x, u)$

(ii) *For* $K > m + 1$

$$\int_X tr\ E_i(t) \sim \sum_{j=0}^K p.f.\ a_{j/2}t^{-(m+1)/2+j/2} - 1/2\left[\int_N a_{(m+1)/2}(1,x)\right] \log t + \psi_i(N),$$

where

$$\psi_i(N) = 1/2\left[\int_1^\infty u^{-1} \int_N tr\ \mathscr{E}_i(1,x,1,x,u) + \int_0^1 u^{-1}\mu_K(u) + \sum_{\substack{j \le K \\ j \neq m+1}} \frac{1}{(m+1)/2 - j/2} \int_N a_{j/2}(1,x)\right].$$

Note the appearance of $\log t$ which is absent in the smooth case. Note also that $\psi_i(N)$, the contribution to the constant term coming from the singularity, is formally equal to the constant term in the Laurent expansion of

$$\int_N tr\ \Delta^{-s}(1,x,1,x), \qquad [5.5]$$

at $s = 0$.

In order to make the expression on the right-hand side of *(ii)* more explicit, we will calculate $a_{j/2}(1,x)$, $\psi_i(N)$ as spectral invariants of N. The calculation proceeds by establishing a connection between $tr\ \mathcal{E}(t)$ and $tr\ \Delta^{-s}$ similar to that which holds in the compact case. Some care must be taken because $C(N)$ is, in fact, not compact. $tr\ \Delta^{-s}$ can be dealt with more easily than $tr\cdot\mathcal{E}(t)$ via **4.4** and the corresponding formulas for types 2–4. As in **4.4**, let the meromorphic function $\psi_i(s)$ be defined by

$$\psi_i(s) = \frac{1}{2\sqrt{\pi}} \sum_{\mu_j>0} \frac{\Gamma(\nu_j - s + 1)}{\Gamma(\nu_j+s)} \Gamma(s - 1/2) \qquad [5.6]$$

where the μ_j run over the eigenvalues corresponding to co-exact i-forms. Set

$$\psi_{i,>}(s) = \frac{1}{2\sqrt{\pi}} \sum_{\nu_j>m+1} \frac{\Gamma(\nu_j - s + 1)}{\Gamma(\nu_j + s)} \Gamma(s - 1/2). \qquad [5.7]$$

Interpret $\psi_i(s)$ as $\equiv 0$ for $i \notin 0, \ldots m - 1$ and the ith Betti number b_i of N as zero if $i \notin 0, \ldots m$. Let B_j denote the jth Bernoulli number.

THEOREM 5.2 *(compare ref.* 10).

(i) $\int_N a_{j/2}(1,x)$

$$= \operatorname*{Res}_{s = (m+1)/2 - j/2} [\psi_{i,>}(s) + 2\psi_{i-1,>}(s) + \psi_{i-2,>}(s)]$$

(ii) $$\psi_i(N) = \tfrac{1}{2} \frac{d}{ds} [s(\psi_i(s) + 2\psi_{i-1}(s) + \psi_{i-2}(s))]|_{s=0}$$
$$+ \tfrac{1}{4}(|\alpha_i| b_i + |\alpha_{i-1}| b_{i-1}) + \tfrac{1}{2} \frac{\alpha_{i-1}}{|\alpha_{i-1}|} b_{i-1}$$
$$+ \operatorname*{Res}_{s=0} \nu^{-1-2s}(i-1)$$

(iii) $$\frac{d}{ds}\left[s\psi_j(s)\right]\Big|_{s=0} = \frac{d}{ds}\left(\frac{s \cdot \Gamma(s - 1/2)}{2\sqrt{\pi}} \nu^{1-2s}\right)\Big|_{s=0}$$
$$+ \operatorname*{Res}_{s=0} \sum_{j=1}^{j\le m/2+1} (-1)^{j-1} \frac{B_j}{j} \nu^{-1-2j-2s}.$$

Note that $\psi(s)$ has at most a simple pole at $s = 0$. Thus, $d/ds[s \cdot \psi(s)]|_{s=0}$ is just the constant term in the Laurent expansion of $\psi(s)$ at $s = 0$. The residues in *i–iii* can be calculated explicitly in terms of residues of the zeta functions (equivalently, coefficients in the asymptotic expansion of $tr\ e^{-\bar{\Delta}t}$) for N. This will be done elsewhere. That precise form of the results in *Theorem 5.2* depends on the fact that the zeta functions on N have simple poles. Because of the $\log t$ in i of *Theorem 5.1*, the zeta functions for X can have double poles. Also, $\Psi_0(N)$ is definitely *not a locally computable invariant.*

6. The Gauss Bonnet formula

In order to apply, the heat equation method to the calculation of the L^2-Euler characteristic $\hat{\chi}$, we must take the alternating sum with respect to i, of the right-hand sides of *Theorem 5.2, ii* (11–15). This has the effect of making the global spectral invariants cancel and we are left with just

$$\hat{\chi}(X) = \int_X P_\chi(\Omega) + \sum_{i=0}^{m-1} (-1)^{i+1} \operatorname*{Res}_{s=0} \nu^{-1-2s}(i)$$
$$+ \tfrac{1}{2}\{\chi[C^*_{0,1}(N)] + \hat{\chi}[C^*_{0,1}(N), N]\}. \qquad [6.1]$$

Here $P_\chi(\Omega)$ denotes the Chern–Gauss Bonnet form. Moreover, given **6.1**, it is easy to see that the same formula holds with $\hat{\chi}$ replaced by χ throughout. The resulting expression, is then equivalent to a Gauss Bonnet formula for the manifold with boundary M but with the boundary term expressed as a spectral rather than a curvature invariant.

7. The signature and η-invariant

When the heat equation method is applied to the signature complex, an analysis like that of the previous two sections yields the formula

THEOREM 7.1.

$$\mathrm{Sig}(X) = \int_X P_L(\Omega) + \eta(N)$$

where $P_L(\Omega)$ *is the characteristic form corresponding to the L-class and* $\eta(N)$ *is the* η*-invariant of Atiyah-Patodi-Singer* (16).

Theorem 7.1 is of course equivalent to their formula because the integral on the right-hand side need only be taken over M,$[P_L(\Omega) \equiv 0$ on $C_{0,1}(N)]$ and

$$\mathrm{Sig}\ X = \mathrm{Sig}(M) \qquad [7.1]$$

(see *Theorem 3.4*). However, the point here is not to give rederivation of their result. Rather I wish to emphasize that *Theorem 7.1* is the natural signature formula for a class of compact singular spaces that are in themselves interesting geometric objects.

Theorem 7.1 has a generalization to the *p.l.* case which seems of particular interest and will be described elsewhere.

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